Fundamentals of MATHEMATICS

FOR JEE MAIN AND ADVANCED

COORDINATE GEOMETRY

Sanjay Mishra

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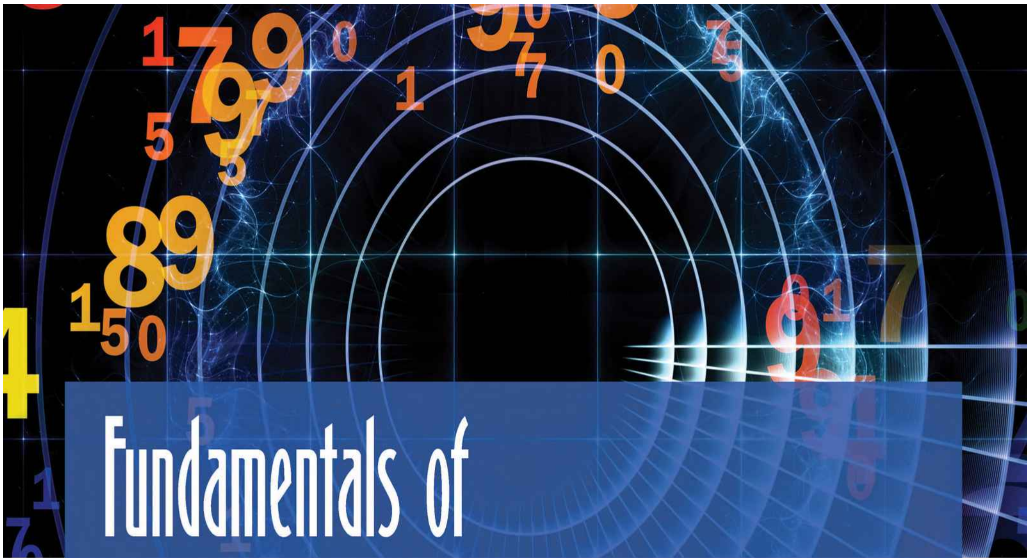

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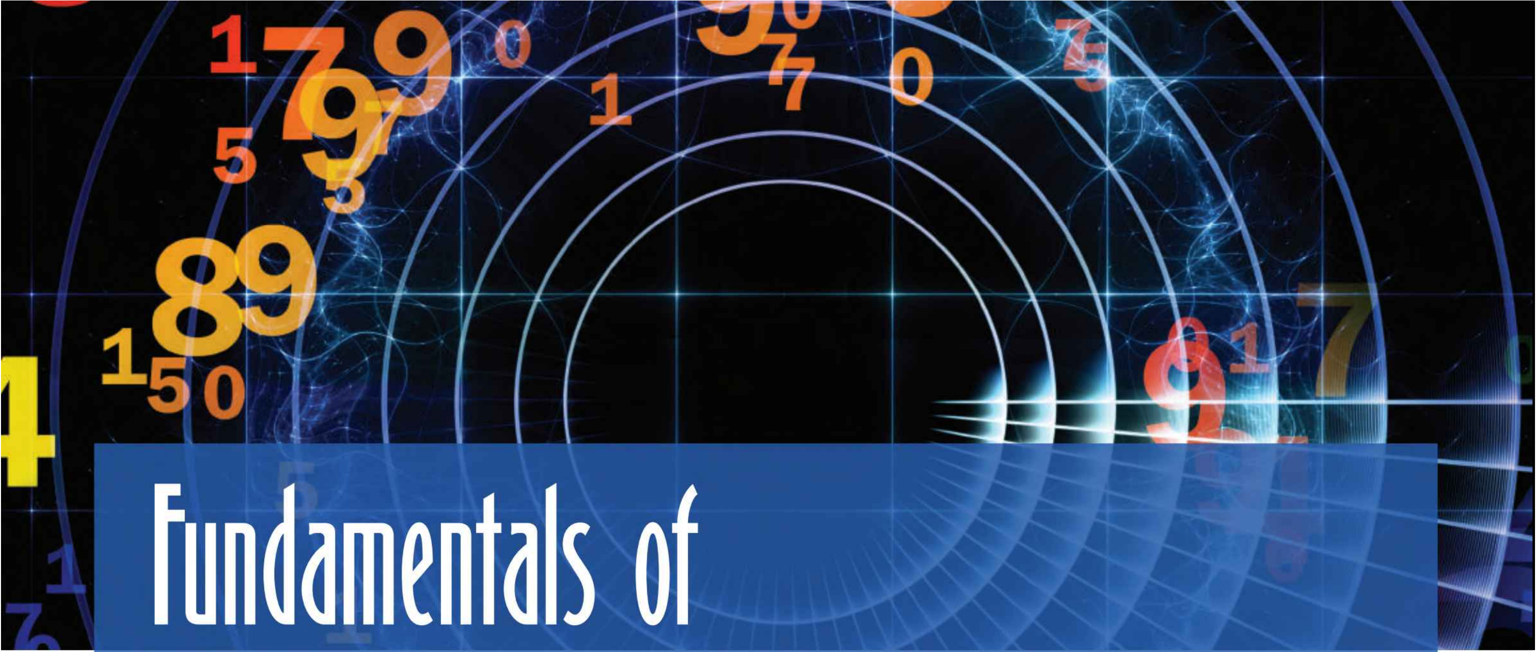

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Fundamentals of Mathematics

Co-ordinate Geometry

Second Edition

Sanjay Mishra

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Preface

Man always has been curious about everything around him, like various shapes of hills, rocks, caves, trees, etc. He also divided lands for cultivation and other purposes. He observed the various-shaped objects and gradually started to draw figures of those objects. In this process, he needed to learn about the dimensions of the things around. This need gave rise to a new branch of mathematics, 'Synthetic Geometry'. The word 'Geometry' is a combination of two Greek words, *geo* (earth) and *metry* (measure).

Synthetic geometry can be regarded as a game played by axiomatic rules created by the ancient Indians and the Greeks. The noted Greek mathematician Euclid and his predecessors as well as his disciples were quite convinced about certain rules. They thought those these rules adequately represented the laws of the real world around them. From this period to the 17th century, only geometric logics and arguments were applied in the study of geometry, which is now familiar in the name of 'synthetic geometry'. It is a study of various theorems, their proofs, solving problems with the help of these theorems and different construction techniques. Although this concept was noble and quite useful in solving problems related to straight lines, triangles, circles and their properties, but was not equally convenient for the study and derivation of the properties of many other curves like, ellipse, parabola, hyperbola, and so on.

In the 17th century-geometry was associated with algebra and many problems of synthetic geometry were conveniently solved applying the concepts of algebra. The study of geometry with the help of algebra is called 'analytical geometry'. The systematic use of algebra in geometry was first carried out by a famous French philosopher René Descartes (1596–1650) in his book *La Géométrie* (published in 1637).

'Co-ordinate Geometry' is defined as the branch of mathematics which includes the study of different curves and figures by representing points in a plane by ordered pairs of real numbers called 'Cartesian Co-ordinates' and it also includes representation of lines and other specific types of curves by algebraic equations in the variables x and y or x, y and z. The notion of co-ordinates lead to the idea of locus and consequently, laid open the path for mathematical analysis, conceptualization of functions, their limits, continuity, differentiability, and ultimately culminated into the advent of the modern form of calculus.

Co-ordinate geometry has a major share in the syllabus of IIT-JEE and other competitive examinations, so its in-depth analysis is important. During my high school days, as an IIT-JEE aspirant, and later, as a mathematics tutor, Mathematics for last fifteen years, I always realized the need of a comprehensive textbook for this subject. I, therefore, always had an insatiable desire to write one.

This book has been written with the objective of providing a textbook as well as an exercise book, focusing on problem solving. I feel, this will not only fulfil the need of a beginner, pre-college student (i.e., students of XI and XII standards), but also meet the requirements of the advanced level students who are preparing for various entrance examinations like IIT-JEE, AIEEE, BIT-SAT, and other state engineering entrance examinations. This book, *Fundamentals of Mathematics—Co-ordinate Geometry*, develops a deep insights into topics, such as Points and Cartesian System, Straight Lines and Family of Straight Lines, Circles and Family of Circle, Parabola, Ellipse and Hyperbola. I personally experienced in my teaching career that the last three conic sections are the most scary from a student's point of view, but highly scoring topics of mathematics as far as competitive exams are concerned. One of the reasons for the phobia in the students' minds against these topics is non-familiarity with these curves in basic classes and lack of good books that lay down these concepts in a student-friendly and lucid manner.

The well-arranged content list will help students and teachers to conveniently access the chapters and sub-topics of their interest. Each chapter is divided into several topics. Each topic contains theory and sometimes sub-topics with sufficient number of worked-out illustrative problems. Students can develop applicative ability of the concepts learned. This is followed by a textual exercise of both objective and subjective type problems, as per the requirements. At the end of the theory of each chapter, a large set of solved examples of both objective and subjective types is given. This will involve application of all the concepts learnt in the chapter so that students can develop mastery over the chapter. The tutorial exercise given at the end contains a large number of multiple-choice problems with single and multiple correct answers, comprehension passages, column matching problems, numerical integer type questions to facilitate the students to do thorough revision of the entire chapter and to enhance their level of understanding of the topics. For teachers, this text book will be quite helpful as it will provide a set of well-graded problems and well-arranged topics that can be used to give home assignments to their students.

All suggestions for improvement are welcome and shall be gratefully acknowledged.

—Sanjay Mishra

Acknowledgements

I am grateful to Pearson Education for keeping faith in me and providing me with an opportunity to transform my yearning, my vast experience of years of teaching, and my knowledge comprehensively into the present textbook, *Fundamentals of Mathematics—Co-ordinate Geometry*. I would like to thank all my friends and teachers, for their valuable criticism, support and advice that was helped me to carve out this work. I am pleased to award special acknowledgements to all my pupils. During my interactions with my students, I received much of the inspiration of writing this textbook. I feel that by interacting with them, I have learnt much more than I could have ever taught them. I wish to thank my parents and all my family members, for their patience and support in bringing out this book and contributing their valuable time for this cause. I extend my special thanks to my team, especially to my assisting teachers, managers, and computer opera for their hard work and dedication in completing this task. I extend my special thanks to my team, Rakesh Gupta, Sanket Sinha, and the DTP Operators at MIIT (EDU.) SERVICES PVT. LTD., for their hard work and dedication in this task.

—Sanjay Mishra

Point and Cartesian System

CHAPTER 1

INTRODUCTION

The word Geometry is a combination of two Greek words *Geo* (earth) and *Metry* (measure). It can be defined as a branch of mathematics which developed to facilitate the study and measurement of various land form. A famous Greek mathematician Euclid (300 BC) in his first systematic treaty on geometry mentioned some axioms and postulates and regarded them as the rules that adequately represent the laws of real world around them. The set of these laws and their applications are together called as Synthetic Geometry. Some of the postluates of Euclidean Geometry are mentioned below.

POSTULATES OF EUCLIDEAN GEOMETRY

1. **Notion of point:** Point is a dimensionless hypothetical object. It is a geometric construction with no dimension.

   However, it is noted that howsoever small dot is placed on a piece of paper, it is theoretically not a point but a combination of infinitely many points. But practically small dots are considered to be points as their all dimensions are reasonably small enough to be ignored and taken as zero. Moreover, we study these ideas in relative sense not in absolute sense, e.g., a dot with area 10^{-7} square cm can be considered as a point in comparison to a circular field with area few square meters but a circle with area 10 cm^2 cannot be treated so, whereas the same circle with 10 cm^2 area can be regarded as a point with respect to plane fields expanding over square miles.
2. **Line:** It is a locus of a point defined as one dimensional curve, it has only length, no breadth and thickness. Lines can be classified in two ways: (i) curved line; (ii) straight line.

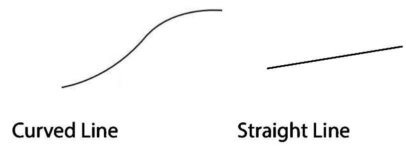


FIGURE 1.1

3. **Idea of intermediacy:** Between any two distinct points on a straight line there always lies another point howsoever close they are.
5. **Line segment and rays:** If a line is truncated by two points (say A and B), then it is known as '*line segment*'. Clearly, a line segment has an initial point (A) and a final point (B). Therefore, has a fixed finite length. When the final point B lies at infinity, then the semi-infinite length line segments obtained are called 'rays'.
6. **Surface:** It is a two-dimensional construction, can be termed as a locus of a line, i.e., having length and breadth, but no thickness. Practically, it has thickness as small as compaired to other two dimensions that it can be ignored.

   Surfaces may be 'plane' as well as 'curved'. If we choose two arbitrary points (A and B) on a surface and joint them by a straight line segment (AB), and every point on the line segment is contained in the surface, then it is regarded as Euclidean plane surface, otherwise known as curved surface.

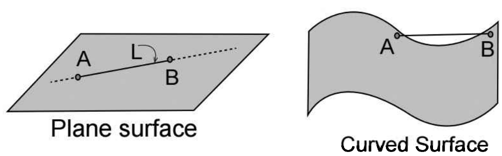


FIGURE 1.2

A straight line L divides an Euclidean plane (P) into two disjoint half planes P_1 and P_2 such that the plane (P) is union of three disjoint set of points P_1, P_2 and L, where L is the set of points on the line.

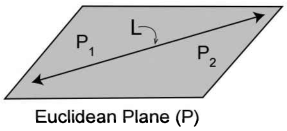


FIGURE 1.3

7. **Solid:** A geometrical construction having three dimensions, obtained by translation or rotation of surfaces is called '*solid*'.

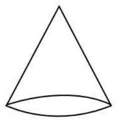


FIGURE 1.4

Frame of Reference

Several methods have been developed by mathematicians to uniquely locate the position of a point in the space. So that various useful observations can be conveniently made with the help of set of fixed points/lines/surfaces.

A set of fixed points/lines/surfaces with respect to which observations are made is called frame of reference.

CO-ORDINATE SYSTEMS

In the 17th centuary AD, the synthetic geometry was associated with algebra and the study of geometry with the systematic application of algebra was first carried out by a famous french Philospher Rene Descartes (1596-1650) in his book *La Geometrie* (published in 1637). The study of geometry with help of algebraic equations is called analytical geometry/Cartesian co-ordinate geometry.

Co-ordinate geometry begins with the study of the *Concept of Point.* Descartes established a relationship between the basic geometric concept of point and the ordered pair of real number (x, y) (where x represents the horizontal distance of the point from origin and y represents the vertical distance of the point from the origin) and this relationship is called Cartesian system of co-ordinates. He successfully explained that any point in the Euclidean plane can be associated with a unique ordered pair (x, y), and thus, the set of all points in Euclidean plane has one to one correspondence with the set of ordered pairs represented by Cartesian product $X \times Y$ ($\mathbb{R} \times \mathbb{R}$). This is the reason why Euclidean plane is also called Cartesian plane or $\mathbb{R} \times \mathbb{R}$ plane or $\mathbb{R}^2$ plane.

The most commonly used frame of references (co-ordinate systems) are of the following three types:

(1) Rectangular co-ordinate system
(2) Oblique co-ordinate system
(3) Polar co-ordinate system

Rectangular Co-ordinate System

It consists of two mutually perpendicular lines intersecting at O called origin as shown below.

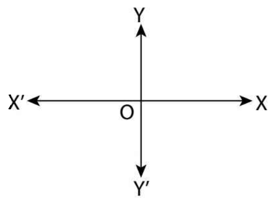


FIGURE 1.5

- Origin is the point from where the observations are made with the help of two axes and a suitably chosen sign convention.
- Horizontal line $X'OX$ is called x-axis (*abscissa axis*).
- Vertical line $Y'OY$ is called y-axis (*ordinate axis*).

Sign Convention

All the distances measured towards OX and OY directions are cosidered to be positive whereas towards OX' and OY' directions are termed negative and any distance measured from origin to any other direction than the above four is taken always positive. The angle measured in clockwise direction is considered negative, whereas that measured in counterclockwise sense is taken as positive angle.

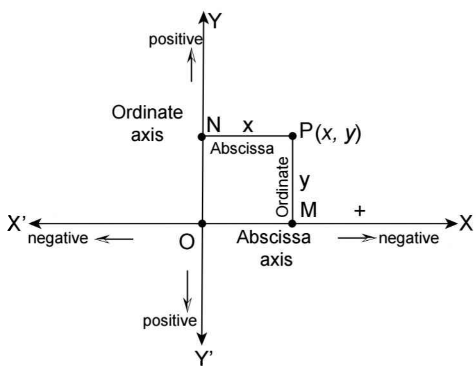


FIGURE 1.6

- **Representation of point:** Any point P in x-y plane can be represented by unique ordered pair of two real numbers x and y as (x, y) and it is defined as co-ordinates of the point P.
- Here x is *abscissa* of point (OM or PN).
- y is *ordinate* of point (ON or PM).
- Therefore, the x-y plane (Cartesian plane) is algebraically represented as Cartesian product of two sets of real numbers, so it is called $\mathbb{R} \times \mathbb{R}$ plane or $(\mathbb{R})^2$ plane. where $\mathbb{R} \times \mathbb{R} = \{(x, y) : x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$.

$\mathbb{R}^+ \times \mathbb{R}^+ =$ set of all points in the 1st quadrant

$\mathbb{R}^+ \times \mathbb{R}^- =$ set of all points in the 4th quadrant

$\mathbb{R}^- \times \mathbb{R}^+ =$ set of all points in the 2nd quadrant

$\mathbb{R}^- \times \mathbb{R}^- =$ set of all points in the 3rd quadrant

REMARKS

(i) *The ordinate of every point on x-axis is* 0, *that is why equation of x-axis is* $y = 0$.

(ii) *The absicissa of every point on y-*axis *is* 0, *that is why equation of y-axis is* $x = 0$.

(iii) *The abscissa* and *ordinate of the origin O are both zero, i.e., (0, 0).*

(iv) *The abscissa* and *ordinate of a point is its algebraic length of perpendicular distance from y-axis* and *x*-axis, *respectively.*

(v)

Quadrants	***XOY (I)***	***X′OY (II)***	***X′OY′(III)***	***(XOY′(IV)***
Sign of x co-ordinates	+	–	–	+
Sign of y co-ordinates	+	+	–	–
Sign of (x, y)	(+, +)	(–, +)	(–, –)	(+, –)

ILLUSTRATION 1: Locate (1, 2), (–1, 2), (–1, –2) and (1, –2) in rectangular co-ordinate system and then find the area of figure obtained by joining them with straight line segments in the given order.

SOLUTION: Clearly, the figure obtained is rectangle ABCD with area 8 square unit.

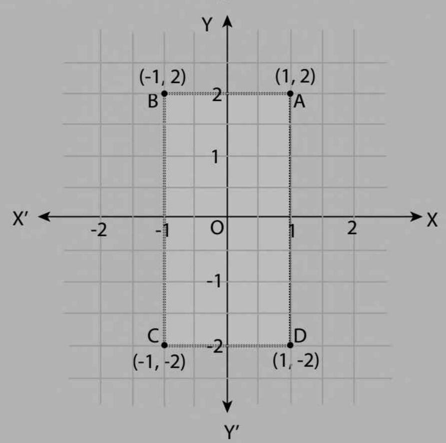


FIGURE 1.7

ILLUSTRATION 2: Locate the position of following points in the correct quadrant.

(a) $(a\alpha^2 + b\alpha + c, b^2 - 4ac)$ $\forall \alpha \in \mathbb{R}$ when $a > 0$ and $b^2 < 4ac$

(b) $(\cos\theta + \sin\theta, \cos\theta - \sin\theta)$ when $\theta \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$

SOLUTION: (a) Given, $b^2 < 4ac \Rightarrow b^2 - 4ac < 0$

Therefore, expression $a\alpha^2 + b\alpha + c$ has same sign as its leading co-efficient.

$\Rightarrow$ Abscissa of point, i.e., $x = a\alpha^2 + b\alpha + c > 0 \ \forall \alpha \in \mathbb{R} \ \because a > 0$ and ordinate of point i.e., $y = b^2 - 4ac < 0$. Clearly, the point lies in IVth quadrant.

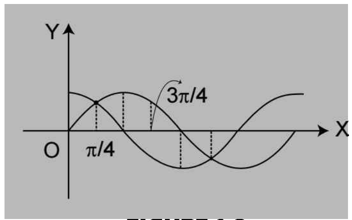


FIGURE 1.8

(b) Consider the co-ordinate of the point $x = \cos\theta + \sin\theta$; $y = \cos\theta - \sin\theta$

As $\forall \theta \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$; $\sin\theta > \cos\theta \Rightarrow \cos\theta - \sin\theta < 0 \Rightarrow y < 0$

whereas $\cos\theta + \sin\theta$ is positive if $\theta \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$.

$\Rightarrow$ $x > 0$. Hence, point lies in the IVth quadrant.

ILLUSTRATION 3: Find the quadrant where the following points are located.

(a) $\left(\cos 10° - \cos 13°, \tan\frac{\pi}{9} - \tan\frac{\pi}{7}\right)$

(b) $(\cos 18° - \sin 15°, \cos 105° - \tan 15°)$

(c) $(\sec^2 9° - \tan^2 9°, \cos 17° - 1)$

(d) $\left(\log_2\left(\frac{x}{x^2+1}\right), \log_{1/2}(x^2+x+1)\right) \forall x \in \mathbb{R}^+$

SOLUTION: (a) Since $\cos x$ decreases with increase of $x \ \forall \ x \in (0, \pi)$ thus $\cos 10° > \cos 13°$

$\Rightarrow x = \cos 10° - \cos 13° > 0$

and as $\tan x$ increases with x in first quadrant and

$\because \frac{\pi}{9} < \frac{\pi}{7} \Rightarrow \tan\frac{\pi}{9} < \tan\frac{\pi}{7} \Rightarrow y = \tan\frac{\pi}{9} - \tan\frac{\pi}{7} < 0$

Thus, $x > 0$ and $y < 0$, so the point lies in 4th quadrant.

(b) Since $\sin\theta$ is an increasing function in the interval $[0, \pi/2]$ and $\cos 18° = \sin 72°$

$\Rightarrow \sin 72° > \sin 15° \Rightarrow \cos 18° - \sin 15° > 0$

Now, $\cos 105° - \tan 15° = \cos(90° + 15°) - \tan 15°$

$= -\sin 15° - \tan 15° < 0$ ($\because$ $\sin 15°$, $\tan 15°$ are both positive)

$\therefore$ Abscissa of point is positive and ordinate negative, consequently the point lies in the IVth quadrant.

(c) Since $\sec^2 9° - \tan^2 9° = 1$ (as $\sec^2\theta - \tan^2\theta = 1$) and $\cos 17° - 1 < 0$ as $\cos\theta \le 1 \ \forall \ \theta \in \mathbb{R}$

$\therefore$ Abscissa of point is positive, whereas the ordinate is negative.

$\Rightarrow$ point lies in IVth quadrant.

(d) Let $y = \frac{x}{x^2+1}$; $x \in \mathbb{R}^+ \Rightarrow x^2y - x + y = 0$, but $x \in \mathbb{R}^+ \Rightarrow D \ge 0 \Rightarrow 1 - 4y^2 \ge 0$

$\Rightarrow \frac{-1}{2} \le y \le \frac{1}{2}$. But $x \in \mathbb{R}^+$, therefore y should be positive. $\Rightarrow \frac{x}{x^2+1} \in \left(0, \frac{1}{2}\right]$

Alternatively, $y = \frac{x}{x^2+1} = \frac{1}{x + \frac{1}{x}}$; $x \in \mathbb{R}^+$

by AM ≥ GM

$$\Rightarrow \frac{x+\frac{1}{x}}{2} \geq 1 \Rightarrow \frac{1}{x+\frac{1}{x}} \in \left(0, \frac{1}{2}\right]$$

$\Rightarrow \log_2\left(\frac{x}{x^2+1}\right) < 0$ as $\log_b a < 0$ for a and b on opposite side of 1. Thus, abscissa is negative.

Now, $x^2 + x + 1 = x^2 + x + 1/4 + 3/4$

$$= \left(x+\frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4} \Rightarrow x^2 + x + 1 \in \left[\frac{3}{4}, \infty\right)$$

But, for $x \in \mathbb{R}^+$, $x^2 + x + 1$ is an increasing function and $x^2 + x + 1 \in (1, \infty)$.

$\Rightarrow \log_{1/2}(x^2 + x + 1) < 0 \Rightarrow$ ordinate is also negative $\Rightarrow$ Point lies in the IIIrd quadrant.

■ OBLIQUE CO-ORDINATE SYSTEM

It consists of two axes which are not perpendicular, i.e., inclined at certain angle ω (ω ≠ 90°) as shown in the figure. That is why the system is known as oblique co-ordinate system. The concept of assigning co-ordinates and sign convention is the same as the rectangular co-ordinate system. The point P with co-ordinates (x', y') means $OM = x'$ and $PM = y'$.

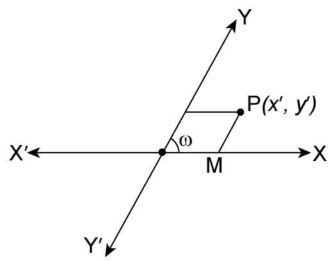


FIGURE 1.9

Relation between rectangular co-ordinate system and oblique co-ordinate system: Consider that a point P having its rectangular co-ordinates (x, y) and oblique co-ordinates (x', y') as shown in the figure below.

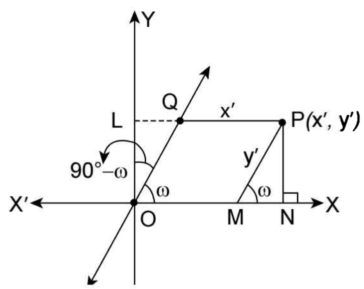


FIGURE 1.10

Now, $x = LP = LQ + QP = y' \cos\omega + x'$ and $y = PN = y' \sin\omega$, Therefore, $x = x' + y' \cos\omega$ and $y = y' \sin\omega$.

ILLUSTRATION 4: Let $P(2,3)$ in a rectangular co-ordinate system. Then, obtain this point with respect to oblique co-ordinate system, where angle between two axes is 60°.

SOLUTION: $x = x' + y' \cos \omega$

$\Rightarrow 2 = x' + y' \cos 60°$ .... (i)

and $y = y' \sin \omega$

$\Rightarrow 3 = y' \sin 60° \Rightarrow y' = 2\sqrt{3}$

∴ From (i), we get $x' = 2 - 2\sqrt{3}\,(1/2) = 2 - \sqrt{3}$

∴ Oblique co-ordinates of point P are $(2 - \sqrt{3}, 2\sqrt{3})$.

Polar Co-ordinate System

This system has its frame of reference that consists of a point O (called as *Pole*) and a Ray (OX) originating from pole known as *initial line*.

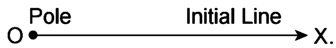


FIGURE 1.11

The line joining any point P to pole (O), i.e., OP ($OP = r$) is called '*radius vector*' and the angle $\angle XOP = \theta$ that radius vector subtends with initial line in anti-clockwise sense is called *vectorial angle*. Position of any point P lying in the plane containing O and initial line OX, can be located uniquely by an ordered pair (r, θ) which is also called the *polar co-ordinates* of point P.

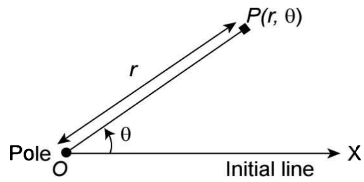


FIGURE 1.12

REMARKS

(i) *Replacing r by –r, the position of point (r, θ) gets reflected in pole (origin). This process is called 'plunging of points'. So (r, θ)* and *(– r, θ) are mirror images of each other in pole.*

(ii) *Adding* 2π *or* $360°$ *(or any integral multiple of* 2π *or* $360°$*) to the vectorial angle does not alter the final position of revolving line so that* (r, θ) *is always the same point as* $(r, \theta + 2n\pi$ *or* $n \times (360°)$*, where* $n \in \mathbb{Z}$.

(iii) *Adding* π *or* $180°$ *or any odd multiple of* π *to the vectorial angle* and *changing the sign of radius vector gives the same point as original. Thus the point* (r, θ) *is same as* $(-r, \theta + \pi)$ *or* $(-r, \theta + (2n + 1)\pi)$.

ILLUSTRATION 5: Locate the points having polar co-ordinates $P\left(2, \frac{\pi}{3}\right), Q\left(-2, \frac{\pi}{3}\right), R\left(-2, -\frac{\pi}{3}\right)$ and $S\left(2, -\frac{\pi}{3}\right)$ on the plane.

SOLUTION: Point P is at 2 unit distance from O and OP makes $\pi/3$ radians angle with OX.

So $\left(2, \frac{\pi}{3}\right)$ lies at P. Similarly, Q denotes $\left(-2, \frac{\pi}{3}\right)$, R denotes $\left(-2, -\frac{\pi}{3}\right)$ and S denotes its image in pole, i.e., $\left(2, -\frac{\pi}{3}\right)$.

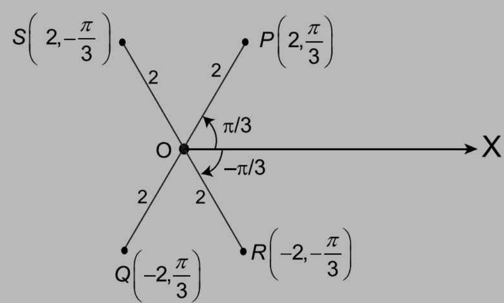


FIGURE 1.13

Relation Between the Polar and Cartesian Co-ordinates

Let $P(x, y)$ be the cartesian co-ordinates with respect to OX and OY and (r, θ) be its polar co-ordinates with respect to pole O and initial line OX. It is clear from from the that

$$OM = x = r\cos\theta \quad \text{.... (1)}$$

and $$MP = y = r\sin\theta \quad \text{.... (2)}$$

Squaring and adding (1) and (2), we get:

$$x^2 + y^2 = r^2 \quad \text{or} \quad r = \sqrt{(x^2 + y^2)}$$

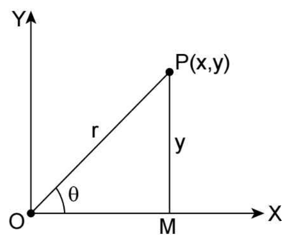


FIGURE 1.14

Dividing (2) by (1), we get $\tan \theta = y/x$.

Now, we have the following cases:

Case I: If P is in the first quadrant, then $\tan \theta = y/x$

$\Rightarrow \theta = \tan^{-1} y/x$

Case II: If P is in the second quadrant, then $\tan \theta = y/x$

$\Rightarrow \theta = \pi - \tan^{-1} |y/x|$

Case III: If P is in the third quadrant, then $\tan \theta = y/x$

$\Rightarrow \theta = \pi + \tan^{-1} |y/x|$ *or* $-\pi + \tan^{-1} |y/x|$

Case IV: If P is in the fourth quadrant, then $\tan \theta = y/x$

$\Rightarrow \theta = -\tan^{-1} |y/x|$ or $2\pi - \tan^{-1} |y/x|$

Therefore, by using $r = \sqrt{x^2 + y^2}$ and the above four cases, we can find the polar co-ordinates of P, when its rectangular co-ordinates are known.

■ DISTANCE BETWEEN TWO POINTS LYING IN A PLANE

1. When Co-ordinates of Two Points are Given in Rectangular Form

Let, $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two given points, then the distance PQ between them is given by.

$$PQ = d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Proof: From Figure 1.15:

$QM = QN - MN = y_2 - y_1$

$PM = ON - OL = x_2 - x_1$

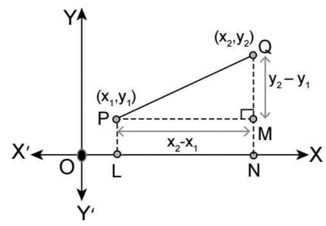


FIGURE 1.15

Now, applying Pythagorous theorem, we have:

$PQ^2 = PM^2 + QM^2$

$\Rightarrow PQ = \sqrt{PM^2 + QM^2}$

$\Rightarrow d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

REMARKS

(i) *When the line PQ is parallel to the y-axis, the abscissa of points P and Q will be equal, i.e.,* $x_1 = x_2$. *Therefore,* $PQ = |y_2 - y_1|$.

(ii) *When the segment PQ is parallel to the x-axis, the ordinates of the points P and Q will be equal, i.e.,* $y_1 = y_2$. *Therefore,* $PQ = |x_2 - x_1|$.

(iii) *The above result holds good even when P and Q lies in the different quadrants.*

ILLUSTRATION 6: Find the distance between points

(a) $P(-2,1)$ and $Q(1, -3)$ (b) $P(5 \tan \theta, 5)$ and origin $(0,0)$.

SOLUTION: (a) $PQ = \sqrt{(-2-1)^2 + (1-(-3))^2} = \sqrt{(-3)^2 + (1+3)^2} = \sqrt{9+16} = \sqrt{25} = 5.$

(b) $PQ = \sqrt{(5\tan\theta - 0)^2 + (5-0)^2} = \sqrt{25\tan^2\theta + 25} = 5\sqrt{\tan^2\theta + 1} = 5|\sec\theta|$

ILLUSTRATION 7: Show that the triangle whose vertices are $A(-3, -4)$, $B(2, 6)$ and $C(-6, 10)$ is right angled.

SOLUTION: We observe that

$AB^2 = (-3-2)^2 + (-4-6)^2 = 125$; $BC^2 = (2-(-6))^2 + (6-10)^2 = 80$

$CA^2 = (-6-(-3))^2 + (10-(-4))^2 = 205$

Thus, $CA^2 = 205 = 125 + 80 = AB^2 + BC^2$ and hence, ABC is right, angled at B.

ILLUSTRATION 8: Prove that the points $A(3, -5)$, $B(-5, -4)$, $C(7, 10)$ and $D(15, 9)$ taken in order are the vertices of a parallelogram.

SOLUTION: Since $AB^2 = (3-(-5))^2 + (-5-(-4))^2 = 65$; $BC^2 = (-5-7)^2 + (-4-10)^2 = 340$

$CD^2 = (7-15)^2 + (10-9)^2 = 65$; $DA^2 = (15-3)^2 + (9-(-5))^2 = 340$

Clearly, opposite sides are equal. So, $ABCD$ is a parallelogram.

ILLUSTRATION 9: If P and Q are two points whose co-ordinates are $(at^2, 2at)$ and $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$, respectively, and S is the point $(a,0)$. Show that $\frac{1}{SP}+\frac{1}{SQ}$ is independent of t.

SOLUTION: We have, $SP = \sqrt{(at^2-a)^2+(2at-0)^2} = |a|\sqrt{(t^2-1)^2+4t^2}$

$$= |a|\sqrt{(t^2+1)^2} = \left|a(t^2+1)\right| = a(t^2+1)$$

and, $$SQ = \sqrt{\left(\frac{a}{t^2}-a\right)^2+\left(-\frac{2a}{t}-0\right)^2} = \sqrt{a^2\frac{(1-t^2)^2}{t^4}+\frac{4a^2}{t^2}}$$

$$= \frac{|a|}{t^2}\sqrt{(1-t^2)^2+4t^2} = \frac{|a|}{t^2}\sqrt{(1+t^2)^2} = \frac{|a|}{t^2}(1+t^2)$$

$$SP = \sqrt{(at^2-a)^2+(2at-0)^2} = |a|\sqrt{(t^2-1)^2+4t^2} = |a|(1+t^2)$$

$$\therefore \frac{1}{SP}+\frac{1}{SQ} = \frac{1}{|a|(t^2+1)}+\frac{t^2}{|a|(t^2+1)} = \frac{1+t^2}{|a|(t^2+1)} = \frac{1}{|a|},$$ which is independent of t.

ILLUSTRATION 10: Find the centre and radius of the circle passing through the vertices of a triangle ABC where the co-ordinates of vertices are given as: $A(0, 1)$, $B(1, 0)$, $C(2, 3)$.

SOLUTION: Let O' be the centre of the circle circumscribing ΔABC.

$(O'A)^2 = (O'B)^2 \Rightarrow (x-0)^2 + (y-1)^2 = (x-1)^2 + (y-0)^2$

$\Rightarrow -2y = -2x \Rightarrow y = x$ ...(i)

$(O'B)^2 = (O'C)^2 \Rightarrow (x-1)^2 + (y-0)^2 = (x-2)^2 + (y-3)^2$

$\Rightarrow 2x + 6y = 12 \Rightarrow x + 3y = 6$ ...(ii)

Solving the equations (i) and (ii), we get $4x = 6 \Rightarrow x = 3/2$

so $y = 3/2$ from equation (i) $\because x = 3/2$

so the centre is $(3/2, 3/2)$

Therefore, the radius of circumcircle $R = \sqrt{(3/2-1)^2+(3/2-0)^2}$

$= \sqrt{1/4+9/4} = \frac{\sqrt{10}}{2}$.

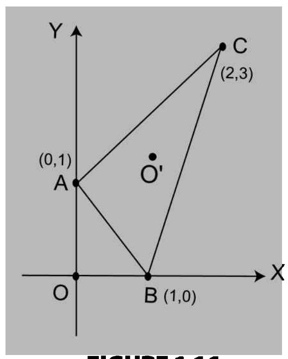


FIGURE 1.16

ILLUSTRATION 11: In any triangle ABC, prove that $AB^2 + AC^2 = 2(AD^2 + DC^2)$, where D is the middle point of BC.

SOLUTION: Take B as origin, BC as the axis of x and a line through B perpendicular to BC as the axis of y. Let $BC = a$, so that C is the point $(a, 0)$ and let A be the point (x_1, y_1). Then D is the point $(a/2, 0)$.

Hencem $AD^2 = \left(x_1 - \frac{a}{2}\right)^2 + y_1^2$ and $DC^2 = \left(\frac{a}{2}\right)^2$

Hencem $2(AD^2 + DC^2) = 2\left[x_1^2 + y_1^2 - ax_1 + a^2/4 + a^2/4\right]$

$= 2x_1^2 + 2y_1^2 - 2ax_1 + a^2.$

Also, $AC^2 = (x_1 - a)^2 + y_1^2$ and $AB^2 = x_1^2 + y_1^2$

Therefore, $AB^2 + AC^2 = 2x_1^2 + 2y_1^2 - 2ax_1 + a^2$

Hence, $AB^2 + AC^2 = 2(AD^2 + DC^2)$.

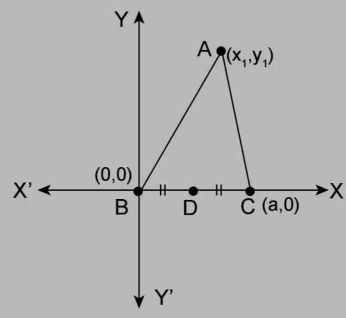


FIGURE 1.17

2. When the Co-ordinates of Points are Given in Oblique System

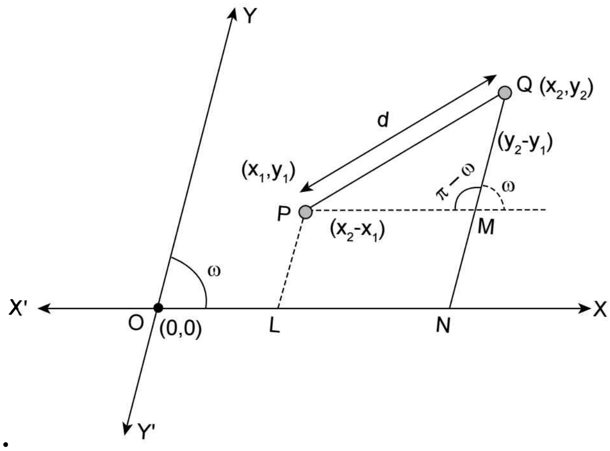


FIGURE 1.18

If the co-ordinate system is oblique, i.e., if the co-ordinate axes are inclined at an angle ω. In this case, the distance between two points P and Q will be given by

$$PQ = d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1)\cos\omega}$$

Hint: (a) To derive the above expression, students are advised to apply cosine formula for $\angle QPM$ in the triangle. PMQ,

i.e., $\cos(\pi - \omega) = \dfrac{(x_2 - x_1)^2 + (y_2 - y_1)^2 - d^2}{2(x_2 - x_1)(y_2 - y_1)}$

(b) If the distance between two points is zero, then both the points have same respective co-ordinates and vice versa.

ILLUSTRATION 12: Let, $A(4,5)$ and $B(2,3)$ be two points in an oblique co-ordinate system, in which the axes are inclined at an angle of $30°$. Then, find the distance AB.

SOLUTION: $AB = \sqrt{(4-2)^2 + (5-3)^2 + 2(4-2)(5-3)\cos 30°}$;

$\Rightarrow AB = \sqrt{2^2 + 2^2 + (2)(2)(2) \times \frac{\sqrt{3}}{2}}$

$= \sqrt{4 + 4 + 4\sqrt{3}}$

$\Rightarrow AB = 2\sqrt{2 + \sqrt{3}}$

3. When the Co-ordinates of Points are Given in Polar Form

Let, $P(r_1, \theta_1)$ and $Q(r_2, \theta_2)$ be two given points in polar co-ordinate system, then the distance PQ between them shall be given as:

$$PQ = d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}$$

Proof: In ΔPOQ, applying cosine rule for the $\angle POQ$,

$$\cos(\theta_1 - \theta_2) = \frac{(OP)^2 + (OQ)^2 - (PQ)^2}{2OP.OQ}$$

$\because OP = r_1$ and $OQ = r_2$

$$\therefore \quad (PQ)^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)$$

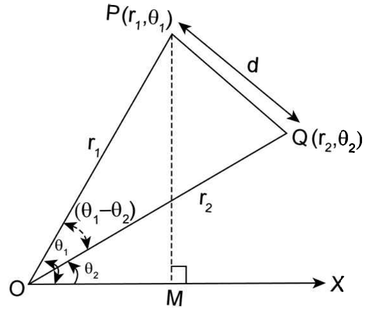


FIGURE 1.19

$$\therefore \quad PQ = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}$$

ILLUSTRATION 13: Find the distance between points $P\left(3, -\frac{\pi}{6}\right)$ and $Q\left(2, \frac{\pi}{6}\right)$.

SOLUTION: $PQ = \sqrt{3^2 + 2^2 - 2\times3\times2\times\cos\left(-\frac{\pi}{6} - \frac{\pi}{6}\right)} \Rightarrow PQ = \sqrt{13 - 12\times\frac{1}{2}} = \sqrt{7}$

ILLUSTRATION 14: (a) Find the polar co-ordinates of the following points (x, y):

(i) $(1, \sqrt{3})$ (ii) $(\sqrt{2}, 1)$ (iii) $(-2, -2)$.

(b) If polar co-ordinates of any points are $(2, \pi/3)$, then find its Cartesian co-ordinates.

SOLUTION: (a) Let the polar co-ordinate of the point be (r, θ), so

(i) $1 = r\cos\theta$ and $\sqrt{3} = r\sin\theta \Rightarrow r = \sqrt{1+3} = 2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

Hence, the polar co-ordinates of the point are $\left(2, \frac{\pi}{3}\right)$.

(ii) $\sqrt{2} = r\cos\theta$ and $1 = r\sin\theta \Rightarrow r = \sqrt{1+2} = \sqrt{3}$ and $\theta = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$,

hence the polar co-ordinates are $\left(\sqrt{3}, \tan^{-1}\frac{1}{\sqrt{2}}\right)$

(iii) Since the cartesian co-ordinate of P are $(-2, -2)$, therefore

$x = -2 = r\cos\theta;\ y = -2 = r\sin\theta \Rightarrow r = \sqrt{4+4} = 2\sqrt{2}$

since the point lies in third quadrant, $\theta = \pi + \tan^{-1}|y/x| = \pi + \tan^{-1}1$

$\Rightarrow \theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$

$\therefore$ The co-ordinates of P are $\left(2\sqrt{2}, \frac{5\pi}{4}\right)$.

(b) Since polar co-ordinates are $(2, \pi/3) \Rightarrow$ the point lies in first quadrant.

$\Rightarrow x = 2\cos\pi/3 = 1$ and $y = 2\sin\pi/3 = \sqrt{3}$

Hence, the cartesian co-ordinates of the point P are $(1, \sqrt{3})$.

ILLUSTRATION 15: Transform the polar equation of the curve $r = a\cos 2\theta$ into Cartesian form.

SOLUTION: Since $x = r\cos\theta, y = r\sin\theta \;\therefore\; x^2 + y^2 = r^2$ ... (i)

and $\cos 2\theta = \cos^2\theta - \sin^2\theta = \left(\dfrac{x^2}{r^2} - \dfrac{y^2}{r^2}\right)$

$\Rightarrow \; a\cos 2\theta = \dfrac{a}{r^2}(x^2 - y^2)$ ...(ii)

$\because \; r = a\cos 2\theta \;\Rightarrow\; r = \dfrac{a}{r^2}(x^2 - y^2) \;\Rightarrow r^3 = a(x^2 - y^2) \;\Rightarrow\; (x^2 + y^2)^{3/2} = a(x^2 - y^2)$

This is the required equation in the Cartesian form.

TEXTUAL EXERCISE–1 (SUBJECTIVE)

1. Find the distance between the points:
(a) $R(a + b, a - b)$ and $S(a - b, -a - b)$
(b) $A(at_1^2, 2at_1)$ and $B(at_2^2, 2at_2)$
(c) $L(a\cos\alpha, a\sin\alpha)$ and $M(a\cos\beta, a\sin\beta)$
(d) $(a\cos\theta, b\sin\theta)$ and $(a\cos\phi, b\sin\phi)$
(e) $(ct_1, c/t_1)$ and $(ct_2, c/t_2)$

2. Find the rectangular co-ordinates of the points whose polar co-ordinates are
(i) $\left(5, \pi - \tan^{-1}\left(\dfrac{4}{3}\right)\right)$
(ii) $\left(5\sqrt{2}, \dfrac{\pi}{4}\right)$

3. If the point (x, y) is equidistant from the points $(a + b, b - a)$ and $(a - b, a + b)$ prove that $bx - ay = 0$.

4. Prove that the points $(2a, 4a)$, $(2a, 6a)$ and $(2a + \sqrt{3}a, 5a)$ are the vertices of an equilateral triangle whose side is $|2a|$.

5. The base of an equilateral triangle with side $2a$ lies along the y-axis such that the mid-point of the base is at the origin. Find the vertices of the triangle.

6. Prove that the distance of the point $(b + a\cos\alpha, c + a\sin\alpha)$ from the point (b, c) is independent of α.

7. Find the distance between the points $(a + r\cos\alpha, b + r\sin\alpha)$ and $(a + r\cos\beta, b + r\sin\beta)$ where r is positive.

8. In which quadrant, do following points lie?
(a) $\left(a^2 + b^2 - ab, 2ab - a^2 - b^2\right): a, b \in \mathbb{R}$ and $a \neq b$
(b) $\left(a + b + c, a^3 + b^3 + c^3 - 3abc\right)$
where a, b, c are different real numbers of same sign.

9. If $P(\cos\theta + \sin\theta, \cos\theta - \sin\theta)$ for $\theta \in [0, \pi/4]$; then find in which quadrant this point P lies.

10. A line segment AB is of length 10 unit, given co-ordinates of $A(2, 3)$ and abscissa of B be 10, then prove that ordinate of B is either –3 or 9.

Answer Keys

1. (a) $2\sqrt{a^2 + b^2}$ (b) $a|(t_2 - t_1)|\sqrt{(t_2 + t_1)^2 + 4}$ (c) $2|a|\sin\left|\left(\dfrac{\alpha}{2} - \dfrac{\beta}{2}\right)\right|$

(d) $\sqrt{a^2(\cos\phi - \cos\theta)^2 + b^2(\sin\phi - \sin\theta)^2}$ (e) $\dfrac{c(t_2 - t_1)}{t_1 t_2}\sqrt{1 + (t_1 t_2)^2}$

2. (i) (–3, 4) (ii) (5, 5)

5. $(\sqrt{3}a, 0), (0, a), (0, -a)$ or $(-\sqrt{3}a, 0), (0, a), (0, -a)$

7. $2r\sin\left(\dfrac{\alpha - \beta}{2}\right)$

8. (a) Fourth quadrant (b) 1st or 3rd quadrant **9.** First quadrant or on x-axis when $\theta = \pi/4$

TEXTUAL EXERCISE–1 (OBJECTIVE)

1. The polar co-ordinates of the point whose cartesian co-ordinates are (–1, –1) is
   (a) $\left(\sqrt{2}, \frac{\pi}{4}\right)$ (b) $\left(\sqrt{2}, \frac{5\pi}{4}\right)$
   (c) $\left(\sqrt{2}, -\frac{\pi}{4}\right)$ (d) $\left(\sqrt{2}, \frac{3\pi}{4}\right)$

2. The cartesian co-ordinates of the point whose polar co-ordinates are $\left(2\sqrt{2}, -\frac{3\pi}{4}\right)$.
   (a) (–2, 2) (b) (2, –2)
   (c) (–2, –2) (d) (2, 4)

3. Cartesian equation of the polar equation $r = a \sin\theta$ is
   (a) $x^2 + y^2 = ay$ (b) $x^2 + y^2 = -ay$
   (c) $x^2 - y^2 = ay$ (d) None of these

4. Cartesian equation is given by $(x - a)^2 + y^2 = a^2$. Then its polar co-ordinate equation is given by
   (a) $x = a(1 + \cos\theta), y = a \sin\theta$
   (b) $x = a(1 - \cos\theta), y = a \cos\theta$
   (c) $x = a(1 + \sin\theta), y = a \cos\theta$
   (d) $x = a(1 - \sin\theta), y = a\cos\theta$

5. Given P' is reflection of P in x-axis. Then the polar co-ordinates of P' in the figure is

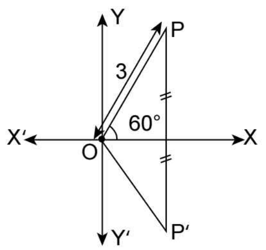


FIGURE 1.20

   (a) $\left(3, \frac{\pi}{3}\right)$ (b) $\left(3, \frac{-\pi}{3}\right)$
   (c) $\left(-3, \frac{-\pi}{3}\right)$ (d) $\left(-3, \frac{\pi}{3}\right)$

6. The Cartesian co-ordinates of the point Q in the figure are

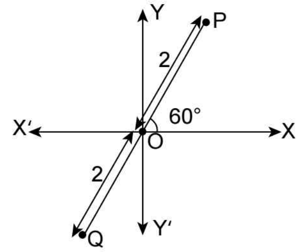


FIGURE 1.21

   (a) $(1, \sqrt{3})$ (b) $(-1, \sqrt{3})$
   (c) $(1, -\sqrt{3})$ (d) $(-1, -\sqrt{3})$

7. The distance between $A(2, 15°)$ and $B(1, 75°)$ is
   (a) $\sqrt{6}$ (b) 6
   (c) $\sqrt{3}$ (d) 3

8. Transformation of polar equation $r = a$ to cartesian equation is
   (a) $x^2 - y^2 = a$
   (b) $x^2 - y^2 = ax$
   (c) $x^2 + y^2 = a^2$
   (d) None of these

9. Three given points which satisfy the given condition $4(PQ)^2 + (PR)^2 = (QR)^2$
   (a) $P(1,2)$, $Q(4,3)$ and $R(2,5)$
   (b) $P(2,2)$, $Q(5,2)$ and $R(3,4)$
   (c) $P(4,2)$, $Q(3,1)$ and $R(4,5)$
   (d) $P(-5,1)$, $Q(1,5)$ and $R(4,7)$

10. The common property of points lying on x-axis, is
   (a) $x = 0$ (b) $y = 0$
   (c) $x = 0, y = 0$ (d) None of these

11. Common property of the bisector of Ist and IIIrd quadrant is
   (a) $y = x$ (b) $y = -x$
   (c) $x^2 = y^2$ (d) None of these

Answer Keys

1. (b) **2.** (c) **3.** (a) **4.** (a) **5.** (b) **6.** (d) **7.** (c) **8.** (c) **9.** (c) **10.** (b)
11. (a)

■ APPLICATION OF DISTANCE FORMULA

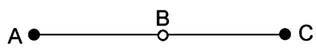
Let A, B, C are three points lying on a plane, then while analysing their relative positions with respect to each other, two conditions arise:

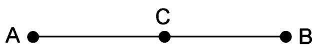
(i) Either they are collinear, or
(ii) They form a triangle

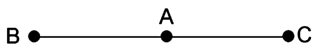
Distance formula can be used to determine whether a set of three points is collinear or forms a triangle as discussed below.

1. Collinearity of Three Given Points

The three given points A, B, C are collinear, i.e., lie on the same straight line, iff the sum of the distances between two points of two pairs is equal to distance between the points of third pair.

i.e., $AB + BC = AC$; 

or $AC + BC = AB$; 

or $BA + AC = BC$; 

2. Formation of Different Triangles

Three points A, B, C form triangles of following types:

(a) **Equilateral triangle:** If $AB = BC = CA$.

(b) **Isosceles triaugle:** If any two sides are equal and their sum is not equal to third side. i.e., their sum is greater than third side.
i.e., $AB = BC$ and $AB + BC > AC$
or $AC = CB$ and $AC + CB > AB$
or $BA = AC$ and $BA + AC > BC$

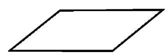
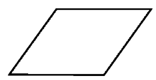
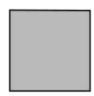
(c) **Scalene triangle:** If all the distances AB, BC, CA are different
i.e., $AB \neq BC \neq AC$, $AC \neq AB$ and A, B and C are non-collinear.

(d) **Right angled triaugle:** If the sum of squares of any two distances is equal to the square of third distance.
i.e., $AB^2 + BC^2 = AC^2$ or $AB^2 + AC^2 = BC^2$ or $AC^2 + BC^2 = AB^2$

3. Formation of Quadrilaterals

Similarly, four points A, B, C, D lying on a plane can form quadrilateral of different types, provided no three of them are collinear.

Quadrilaterals are classified in the table ahead, alongwith their elementary properties.

S. No.	Quadrilateral	Side/Angle	Diagonals and Angle Between Diagonals	Diagram
1	Parallelogram, but not rectangle.	Opposite sides are equal, no angle is right angle.	Diagonals are unequal, but bisect each other.	
2	Rectangle.	A parallelogram having all angles right angles.	Diagonals are equal and bisect each other (*not necessarily at right angle*).	
3	Rhombus, but not square.	All four sides are equal, no angle is right angle.	Diagonals are unequal, but bisect each other at right angle.	
4	Square.	Rhombus all angles having right angles.	Diagonals are equal bisecting each other at right angle.	

REMARKS

(i) *It should be kept in mind that four points A, B, C, D would form a quadrilateral iff no three of the given four points are collinear.*

(ii) *If the co-ordinates of points A* and *B are* (x_1, y_1) and (x_2, y_2), *then the co-ordinates of mid-point of line segment AB would be* $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$. *Students are very much familiar with this formula known as* **mid-point formula** *as they learnt it in their junior classes,* and *it would be exaplained in the article section formula of this chapter.*

ILLUSTRATION 16: Using distance formula, show that the points $P(1, 5)$; $Q(2, 4)$; $R(3, 3)$ are collinear.

SOLUTION: $PQ = \sqrt{(2-1)^2+(4-5)^2} = \sqrt{2}$; $QR = \sqrt{(1)^2+(1)^2} = \sqrt{2}$; $PR = \sqrt{(2)^2+(2)^2} = 2\sqrt{2}$

$\therefore$ $PQ + QR = PR$. **Hence, Proved.**

ILLUSTRATION 17: Prove that the points (2, –2), (–3,8) and (–1,4) are collinear.

SOLUTION: Let the given points be A (2, –2), B(–3, 8) and C(–1, 4), then

$$AB = \sqrt{(2-(-3))^2+(-2-8)^2} = \sqrt{25+100} = \sqrt{125} = 5\sqrt{5}$$

$$AC = \sqrt{(2-(-1))^2+(-2-4)^2} = \sqrt{9+36} = \sqrt{45} = 3\sqrt{5}$$

$$BC = \sqrt{(-3-(-1))^2+(8-4)^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$$

Hence, $AC + BC = AB$. So A, B, C are collinear.

ILLUSTRATION 18: There are four points (0,–1), (6,7), (–2,3) and (8,3). Find out which kind of quadrilateral these points would form.

SOLUTION: Let $A(0, -1)$, $B(6, 7)$, $C(-2, 3)$ and $D(8, 3)$ be the given points. Then,

$$AD = \sqrt{(8-0)^2+(3+1)^2} = \sqrt{64+16} = 4\sqrt{5}\ ;\ BC = \sqrt{(6+2)^2+(7-3)^2} = \sqrt{64+16} = 4\sqrt{5}$$

$$AC = \sqrt{(-2-0)^2+(3+1)^2} = \sqrt{4+16} = 2\sqrt{5} \text{ and } BD = \sqrt{(8-6)^2+(7-3)^2} = \sqrt{4+16} = 2\sqrt{5}$$

$\therefore$ $AD = BC$ and $AC = BD$.

If we take the points as shown in the figure, then we observe that opposite sides are equal, so the figure is a parallelogram. Moreover,

$$AB = \sqrt{(6-0)^2+(7+1)^2} = \sqrt{36+64} = 10$$

$$CD = \sqrt{(8+2)^2+(3-3)^2} = 10.$$

Clearly, $AB^2 = AD^2 + DB^2$ and $CD^2 = CB^2 + BD^2$.

$\Rightarrow$ $\angle ADB = DBC = 90°$. $\Rightarrow$ $ADBC$ is a rectangle.

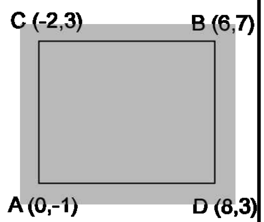


FIGURE 1.22

ILLUSTRATION 19: The triangle OAB is right angled with right angle at O, where points O, A, B are (0, 0), (cosθ, sinθ) and (cosϕ, sin ϕ), respectively, then θ and ϕ are connected by the relation.

(a) $\sin\left(\frac{\theta-\phi}{2}\right) = \pm\frac{1}{\sqrt{2}}$

(b) $\cos\left(\frac{\theta-\phi}{2}\right) = \pm\frac{1}{\sqrt{2}}$

(c) $\cos(\theta-\phi) = 0$

(d) None of these

SOLUTION: Clearly, $OA = OB = 1$ as $\cos^2x + \sin^2x = 1\ \ \forall x \in \mathbb{R}$

$\therefore$ $AB^2 = OA^2 + OB^2 = 2$

or $(\cos\theta - \cos\phi)^2 + (\sin\theta - \sin\phi)^2 = 2$ or $1 + 1 - 2(\cos\theta \cdot \cos\phi + \sin\theta \cdot \sin\phi) = 2$

or $\cos(\theta-\phi) = 0$ or $2\cos^2\left(\frac{\theta-\phi}{2}\right) - 1 = 0$ or $1 - 2\sin^2\left(\frac{\theta-\phi}{2}\right) = 0$

Option (a), (b), (c) are correct.

ILLUSTRATION 20: Find the nature of triangle formed, having vertices (–2, 2), (8, –2), (–4, –3).

SOLUTION: Let $A(-2, 2)$, $B(8, -2)$ and $C(-4, -3)$, then $AB = \sqrt{10^2+4^2} = 2\sqrt{29}$

$BC = \sqrt{12^2+1} = \sqrt{145}$; $CA = \sqrt{4+25} = \sqrt{29}$

Clearly, $AB^2 + AC^2 = BC^2$ (Pythagoras theorem) $\Rightarrow$ Δ is right angled triangle.

ILLUSTRATION 21: Find the number of integer points on x-axis whose distance (p) from the point (2, 3) is such that $p \in (3, 5]$ (Point (x, y) is called integer point if $x, y \in \mathbb{Z}$.)

SOLUTION: Let P be (2, 3) and M is foot of perpendicular from P to x-axis.

$\Rightarrow$ $PM = 3$, Q be geometrically any arbitrary point on x-axis 5 units away from P.

$\therefore$ By Pythagoras theorem $MQ = \sqrt{5^2 - 3^2} = 4$.

$\Rightarrow$ All integer points between (–2, 0) to (6, 0) on x-axis are desired points except for (2, 0).

$\Rightarrow$ number of points = 8

Aliter: Let Q is $(x, 0)$

$\Rightarrow 3 < PQ \le 5 \Rightarrow 3 < \sqrt{(x-2)^2 + 9} \le 5$

$\Rightarrow 9 < (x-2)^2 + 9 \le 25 \Rightarrow 0 < (x-2)^2 \le 16$

$\Rightarrow 0 < |x - 2| \le 4 \Rightarrow x \in [-2, 6]$, but $x \ne 2$

$\Rightarrow$ $x = -2, -1, 0, 1, 3, 4, 5, 6$ number of points is clearly 8.

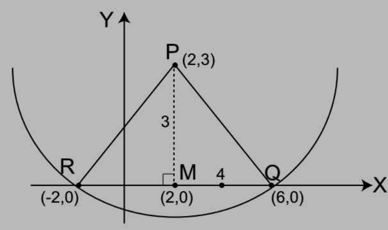


FIGURE 1.23

ILLUSTRATION 22: If $P(1, 2)$, $Q(4, 6)$, $R(5, 7)$ and $S(a, b)$ are the vertices of a parallelogram $PQRS$, then

(a) $a = 2, b = 4$
(b) $a = 3, b = 4$
(c) $a = 2, b = 3$
(d) $a = 3, b = 5$

SOLUTION: $PQRS$ will represent a paralleogram if and only if the mid-point of PR is same as that of QS.

That is, if and only if $\frac{1+5}{2} = \frac{4+a}{2}$ and $\frac{2+7}{2} = \frac{6+b}{2} \Rightarrow a = 2$ and $b = 3$.

ILLUSTRATION 23: The points (– 2, 3), (3, 8) and (4,1) are the vertices of

(a) an isosceles triangle
(b) an equilateral triangle
(c) right angled triangle
(d) None of these

SOLUTION: From figure, it is clear that points are non collinear and hence, form a triangle. In $\Delta\, ABC$,

$AB = \sqrt{(3+2)^2 + (8-3)^2} = \sqrt{50}$;

$BC = \sqrt{(4+2)^2 + (1-3)^2} = \sqrt{40}$

$AC = \sqrt{(3-4)^2 + (8-1)^2} = \sqrt{50}$

$\therefore$ $\Delta\, ABC$ is isosceles

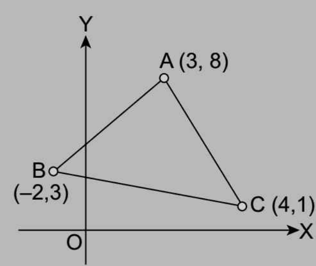


FIGURE 1.24

TEXTUAL EXERCISE–2 (SUBJECTIVE)

1. (a) Prove that the points (1, 1), (–2, 7) and (3, –3) are collinear.
   (b) Check whether the points (0, 8/3), (1, 3) and (82, 30) are the vertices of an isosceles triangle?
2. Show that the point A (0, –1), B (2, 1), C (0, 3) and D (–2, 1) are the vertices of a square.
3. Show that the points (a, a), $(-a, -a)$ and $(-a\sqrt{3}, a\sqrt{3})$ are vertices of an equilateral triangle.
4. If $A \equiv (3, 4)$ and B is a variable point on the lines $|x| = 6$. If $AB \le 4$, then find the number of positions of B with integral co-ordinates.

5. Find the number of integral points on co-ordinate axes which are at a distance c units ($c \le 3$) from (2, 3).

6. If O is the origin and the co-ordinates of any two points Q_1 and Q_2 are (x_1, y_1) and (x_2, y_2) respectively, then prove that $OQ_1 \cdot OQ_2 \cos \angle Q_1 OQ_2 = x_1x_2 + y_1y_2$

7. Show that the points $A(3, 3)$, $B(h, 0)$ and $C(0, k)$ are collinear if $\frac{1}{h}+\frac{1}{k}=\frac{1}{3}$.

8. Find the distance between the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$, where t_1 and t_2 are the roots of the equation $x^2 - 2\sqrt{3}x + 2 = 0$ and $a > 0$.

Answer Keys

1. (b) no 4. 5 points 5. 6 such points exist. 8. $4a(|t_1 - t_2|)$ or $8a$ as $|t_1 - t_2| = 2$

TEXTUAL EXERCISE–2 (OBJECTIVE)

1. The co-ordinates of a point are (0, 1) and the ordinate of another point is –3. If the distance between the two points is 5, then the abscissa of another point is
   (a) 3 (b) –3
   (c) ±3 (d) 1

2. One of the vertices of a square is origin and adjacent sides of the square are coincident with positive axes. If length of side is 5 then which will not be its one of the vertices?
   (a) (0, 5) (b) (5, 0)
   (c) (–5, –5) (d) (0, 0)

3. If the points $A(6,-1)$, $B(1,3)$ and $C(x,8)$ be such that $AB = BC$, then the values of x is
   (a) –3, 5 (b) 3, –5
   (c) –3, –5 (d) 3, 5

4. The distance between the points $(am_1^2, 2am_1)$ and $(am_2^2, 2am_2)$ is
   (a) $|a(m_1 - m_2)|\sqrt{(m_1+m_2)^2+4}$
   (b) $(m_1 - m_2)\sqrt{(m_1+m_2)^2+4}$
   (c) $a(m_1 - m_2)\sqrt{(m_1+m_2)^2-4}$
   (d) $(m_1 - m_2)\sqrt{(m_1+m_2)^2-4}$

5. If the points (0,0), $(2, 2\sqrt{3})$ and (a, b) be the vertices of an equilateral triangle, then the ordered pair (a, b) given as
   (a) $(-2, 2\sqrt{3})$ (b) $(2, -2\sqrt{3})$
   (c) (4, 0) (d) (–4,0)

6. If the co-ordinates of vertices of ΔOAB are (0,0); $(\cos\alpha, \sin\alpha)$ and $(-\sin\alpha, \cos\alpha)$ respectively, then $OA^2 + OB^2$ is equal to
   (a) 0 (b) 1
   (c) 2 (d) 3

7. Given points $A(5,2)$, $B(2,3)$, $C(6,5)$ are vertices of which of the following geometrical figures?
   (a) equilateral triangle
   (b) right angled isosceles triangle
   (c) scalene triangle
   (d) None of these

8. Given points $P(0,0)$, $Q(3, \pi/2)$ and $R(3, \pi/6)$ are the vertices of
   (a) isosceles triangle but not equilateral
   (b) right angled isoscles triangle
   (c) equilateral triangle
   (d) None of these

9. If $A(1,4)$ and $B(4,8)$ are two points, P is a point on AB such that $AP = AB + BP$. If $AP = 10$, then the co-ordinates of P are
   (a) (12, 7) (b) (7, 12)
   (c) (2, –3) (d) (–3, 2)

10. The points $A(a, b)$, $B(a_1, b_1)$ and $C(2a_1 - a, 2b_1 - b)$ are
   (a) vertices of an scalene triangle
   (b) vertices of right angled triangle
   (c) collinear
   (d) vertices of an isosceles triangle

11. The point whose abscissa is equal to its ordinate and which is equidistant from the points (1,0) and (0,3) is
   (a) (1,1) (b) (2,2)
   (c) (3,3) (d) (4,4)

12. If $A(9, -9)$, $B(1, 3)$ are the ends a right angled isosceles triangle, then the third vertex is

(a) $(8, -1)$ (b) $(-8, 2)$
(c) $(8, -8)$ (d) None of these

13. A triangle ABC right angled at A, has points A and B as $(2, 3)$ and $(0, -1)$ respectively. If $BC = 5$ units, then the point C is

(a) $(4, 2)$ (b) $(-4, 2)$
(c) $(0, 4)$ (d) $(3, -3)$

14. If the point $(x, -1)$,$(3, y)$,$(-2,3)$ and $(-3,-2)$ be the vertices of a parallelogram, then

(a) $x = 2, y = 4$ (b) $x = 1, y = 2$
(c) $x = 4, y = 2$ (d) None of these

15. If the three vertices of a rectangle taken in order are the points $(2, -2)$, $(8, 4)$ and $(5, 7)$. The co-ordinates of the fourth vertex is

(a) $(1,1)$ (b) $(1,-1)$
(c) $(-1,1)$ (d) None of these

16. If $P(1, 2)$, $Q(4, 6)R(5, 7)$ and $S(a, b)$ are the vertices of a parallelogram $PQRS$, then

(a) $a = 2, b = 4$ (b) $a = 3, b = 4$
(c) $a = 2, b = 3$ (d) $a = 3, b = 5$

Answer Keys

1. (c)	**2.** (c)	**3.** (a)	**4.** (a)	**5.** (a), (c)	**6.** (c)	**7.** (b)	**8.** (c)	**9.** (b)	**10.** (c)
11. (b)	**12.** (a)	**13.** (a), (c)	**14.** (a)	**15.** (c)	**16.** (c)				

■ SECTION FORMULA (DIVISION OF A LINE SEGMENT BY A POINT)

If a point P lies on the line segment AB such that $PA : PB = m : n \in \mathbb{R}$, then it is said that P divides AB in the ratio $m : n$. If the ratio is positive we say that P divides line segment AB internally in the ratio $m : n$. i.e., P lies between A and B, on the line segment AB.

If the ratio is negative, then we say that P divides line segments AB externally in the ratio $|m|: |n|$ i.e., P lies outside the line segment AB, either on AB produced or on BA produced.

Depending upon the location of point P i.e., in between AB or out side AB section formula is known as section formula for internal division formula or section formula for external division.

(a) Internal Division: Co-ordinates of a point which divides the line segment joining two points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $m: n$, internally are

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$

Proof: From the similar triangles ΔAHP and ΔPKB,

we have $\frac{AP}{PB} = \frac{AH}{PK} = \frac{PH}{BK}$

$$\Rightarrow \frac{m}{n} = \frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y}.$$

$$\Rightarrow x = \frac{mx_2 + nx_1}{m+n}, \quad y = \frac{my_2 + ny_1}{m+n}$$

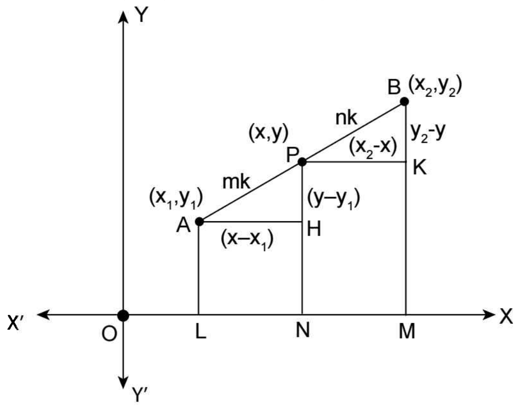


FIGURE 1.25

REMARKS

(i) *The diagram given below helps in memorising the section formula.*

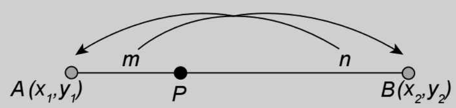


FIGURE 1.26

(ii) *If P is the mid-point of AB, then it divides AB in the ratio 1: 1, so its co-ordinates are* $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.